Black Hole Fluctuations and Backreaction in Stochastic Gravity

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(umdpp 03-20, Oct 4, 2002)

- To appear in a special issue of Foundations of Physics *Thirty Years of Black Hole Physics* edited by L. Horwitz.

Abstract

We present a framework for analyzing black hole backreaction from the point of view of quantum open systems using influence functional formalism. We focus on the model of a black hole described by a radially perturbed quasi-static metric and Hawking radiation by a conformally coupled massless quantum scalar field. It is shown that the closed-time-path (CTP) effective action yields a non-local dissipation term as well as a stochastic noise term in the equation of motion, the Einstein-Langevin equation. Once the thermal Green's function in a Schwarzschild background becomes available to the required accuracy the strategy described here can be applied to obtain concrete results on backreaction. We also present an alternative derivation of the CTP effective action in terms of the Bogolyubov coefficients, thus making a connection with the interpretation of the noise term as measuring the difference in particle production in alternative histories.

This essay is dedicated to my friend and colleague, Professor Jacob Bekenstein, with great admiration and fond memories since our graduate student days. Jacob's pioneering contributions helped establish a new discipline, black hole thermodynamics, and ushered in a new era of research in gravitation, statistical and quantum physics which probes the nature of spacetime and matter at the most fundamental level. – B. L. Hu

1 Introduction

Quantum field theory in curved spacetime [1] is a subject explored from the mid 60's to the 70's which deals with important effects such as cosmological particle creation [2] and Hawking radiation from black holes [3]. The effect of particle creation on the background spacetime – known as the backreaction problem – becomes important when the energy reaches the Planck scale, as in the early universe [4] and at the final stages of black hole evolution [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Investigation of backreaction problems started with the study of the renormalization or regularization of the energy momentum tensor in curved spacetimes in the late 70's. Because of the higher symmetry in cosmological spacetimes, backreaction studies of processes therein have progressed further than the corresponding black hole problems, which to a large degree is still concerned with finding the right approximations for the energy momentum tensor [1] for even the simplest spacetimes such as the spherically symmetric family including the important Schwarzschild metric. Though arduous and demanding, the effort continues on because of the importance of backreaction effects of Hawking radiation in black holes. They are expected to address some of the most basic issues such as black hole thermodynamics [25, 26, 27] and the black hole end-state and information loss puzzles [28].

The most significant developments in the implementation and physical aspects of backreaction problems in the 80's as exemplified first in cosmological spacetimes are perhaps the introduction of an effective action which yields real and causal equations of motion. (For advances in the axiomatic theoretical aspects, see the work of, e.g., Kay and Wald, Flanagan and Wald [29, 30].) This is known as the closed-time-path (CTP) or the Schwinger-Keldysh method [31]. From this one can identify dissipative effects in an unambiguous manner and in the true statistical mechanical sense. In the 90's, the most significant development was perhaps the introduction of quantum open systems concepts [32] and the influence functional method [33], which enables one to identify the origin of quantum noise and recognize the importance of fluctuations. The latter effort has crystallized into a new theory known as stochastic semiclassical gravity (SSG) which is the next stage beyond semiclassical gravity (SCG) towards quantum gravity. These developments were summarized in reviews, (e.g., [34, 35, 36, 37, 38]) to which the reader is referred for details and further references.

Here we wish to address the old black hole backreaction problems with these newer insights. It is not our intention to seek better approximations for the regularized energy momentum tensor, but to point out new ingredients lacking in the existing framework based on SCG. In particular one needs to consider both the dissipation and the fluctuations aspects in the back reaction effects of particle creation or vacuum polarization.

In a short note [39] we discussed the formulation of the problem in this new light, commented on the shortcomings of existing works, and sketched the strategy behind our own approach to the

[1] The latest important work is that of Hiscock, Larson and Anderson [15] on backreaction in the interior of a black hole, where one can find a concise summary of earlier work. To name a few of the important landmarks in this endeavor (this is adopted from [15]), Howard and Candelas [16, 17] have computed the stress-energy of a conformally invariant scalar field in the Schwarzschild geometry. Jensen and Ottewill [18] have computed the vacuum stress-energy of a massless vector field in Schwarzschild. Approximation methods have been developed by Page, Brown, and Ottewill [19, 20, 21] for conformally invariant fields in Schwarzschild spacetime, Frolov and Zel'nikov [22] for conformally invariant fields in a general static spacetime, Anderson, Hiscock and Samuel [13] for massless arbitrarily coupled scalar fields in a general static spherically symmetric spacetime. Furthermore the DeWitt-Schwinger approximation has been derived by Frolov and Zel'nikov[23, 24] for massive fields in Kerr spacetime, Anderson Hiscock and Samuel [13] for a general (arbitrary curvature coupling and mass) scalar field in a general static spherically symmetric spacetime and have applied their method to the Reissner-Nordström geometry [14].

black hole fluctuations and backreaction problem. Here we continue the train of thought with more details for the class of quasi-static black holes, leaving the more demanding dynamical collapse problem to a later exposition. Thus we only address the first set of major issues mentioned above.

From the new perspective provided by statistical field theory and stochastic gravity, it is not difficult to project that backreaction effect is the manifestation of a fluctuation- dissipation relation (FDR) [40]. This was first conjectured by Candelas and Sciama [41] for a dynamic Kerr black hole emitting Hawking radiation. Mottola [42] derived from linear response theory a FDR for a static black hole (in a box) in quasi-equilibrium with its radiation. In [39] we showed the shortcomings of these two important earlier work. In short, Mottola's linear response FDR is based on the assumption of a specified background spacetime (static in this case) and state (thermal) of the matter field(s). But linear response theory is not suitable for backreaction investigations because the spacetime and the state of matter should be determined in a self-consistent manner by their dynamics and mutual influence.

For Candelas and Sciama [41], the classical formula they showed relating the dissipation in area linearly to the squared absolute value of the shear amplitude is suggestive of a fluctuation-dissipation relation. When the gravitational perturbations are quantized (they choose the quantum state to be the Unruh vacuum) they argue that it approximates a flux of radiation from the hole at large radii. Thus the dissipation in area due to the Hawking flux of gravitational radiation is allegedly related to the quantum fluctuations of gravitons. Our criticism is that their's is not a FDR in the truly statistical mechanical sense because it does not relate dissipation of a certain quantity (in this case, horizon area) to the fluctuations of *the same quantity*. To do so would require one to compute the two point function of the area, which, being a four-point function of the graviton field, is related to a two-point function of the stress tensor. The stress tensor is the true "generalized force" acting on the spacetime via the equations of motion, and the dissipation in the metric must eventually be related to the fluctuations of this generalized force for the relation to qualify as a FDR.

From this reasoning, we see that the stress energy bi-tensor and its vacuum expectation value known as the noise kernel, are the new ingredients in backreaction considerations. But these are exactly the centerpiece in stochastic gravity. Therefore the correct framework to address semiclassical backreaction problems is stochastic gravity theory, where fluctuations and dissipation are the equally essential components. The calculation of 4-point functions of the metric perturbations h_{ab} in Minkowski spacetime has been carried out by Martin, Roura and Verdaguer [43, 44], for thermal fields in black hole spacetime and scalar fields in general spacetimes by Campos, Hu and Phillips [45, 46, 47, 48]. Returning to the point of understanding backreaction as manifestation of FDR, a generalized FDR expression relating dissipation (of anisotropy in the case under their study) and fluctuations (measured by particle numbers created in neighboring histories) was obtained by Hu and Sinha earlier [49] for cosmological backreaction problems.

In this note we present a sketch of the stochastic gravity theory and quantum open systems as applied to black hole backreaction problems focusing only on the quasi-static class. Campos and Hu [45] treated the far field case, while here we consider the near horizon case. We use the same approach and show the appearance of two new terms in the stochastic effective action and equation of motion for backreaction: nonlocal dissipation and (generally colored) noise kernels. When the noise average of this Einstein-Langevin equation is taken, York's [7] semiclassical equations for radially perturbed quasi-static black holes is recovered. We can only present the overall structure of the theory and the strategy of our approach, but not the details, because the Green function for a scalar field in the Schwarzschild metric comes only in an approximate form (e.g. Page approximation [19], which, though reasonably accurate for the stress tensor, fails poorly for the noise kernel [47]). In addition we present an alternative way to obtain the CTP effective action, i.e., by expressing it in terms of the Bogolyubov coefficients, which measure not only the number of particles created, but also the difference of particle creation in alternative histories. We see this as a useful avenue to explore the noise and fluctuations issues in black hole physics.

We begin in Sec. 2 with a summary description of quantum open systems followed by a sketch of stochastic gravity theory. In Section 3 we use a simple case of a scalar field in a static, spherically perturbed Schwarzschild spacetime (York's model) to illustrate how to calculate the stochastic effective action and derive the Einstein-Langevin equation. In Sec. 4 we derive an expression for the CTP effective action in terms of the Bogolyubov coefficients. This is an original result not yet published anywhere. We conclude in Sec. 5 with some discussions of the potentials and limitations of our work, and how it relates to other work on black hole fluctuations.

2 Stochastic Approach to Backreaction Problems

2.1 Quantum Open Systems

Here we give a brief schematic summary of the Influence Functional method of analyzing quantum open systems to illustrate the concept, the details can be found in [33]. This approach is designed to deal with the situation in which the system S described , say, by the degrees of freedom x is interacting with an environment E, described by the degrees of freedom q. [2] The full closed quantum system $S+E$ is described by a density matrix $\rho(x,q;x',q',t)$. If we are interested only in the state of the system as influenced by the overall effect, but not the precise state of the environment, i.e, the dynamics of the open system, then the reduced density matrix $\rho_r(x,x',t) = \int dq\ \rho(x,q;x',q,t)$ would provide the relevant information. (The subscript r stands for reduced.) Assuming that the action of the coupled system decomposes as $S = S_s[x] + S_e[q] + S_{int}[x,q]$, and that the initial density matrix factorizes (i.e., takes the tensor product form), $\rho(x,q;x',q',t_i) = \rho_s(x,x',t_i)\rho_e(q,q',t_i)$, the reduced density matrix is given by

$$\rho_r(x,x',t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx'_i \int_{x_i}^{x_f} Dx \int_{x'_i}^{x'_f} Dx'\ e^{i(S_s[x]-S_s[x']+S_{IF}[x,x',t])}\ \rho_r(x_i,x'_i,t_i\ ) \tag{2.1}$$

where S_{IF} is the influence action related to the influence functional $\mathcal{F}$ defined by

$$\mathcal{F}[x,x'] \equiv e^{iS_{IF}[x,x',t]} \equiv \int\ dq_f\ dq_i\ dq'_i \int_{q_i}^{q_f} Dq \int_{q'_i}^{q_f} Dq'\ e^{i(S_e[q]+S_{int}[x,q]-S_e[q']-S_{int}[x',q'])}\rho_e(q_i,q'_i,t_i). \tag{2.2}$$

S_{IF} in general is complex. Retaining only quadratic terms (an approximation which covers many of the interesting applications that we will consider later), we may write

$$S_{IF}(x,x') = \int\ dt\ dt'\ \{\frac{1}{2}(x-x')(t)D(t,t')(x+x')(t') + \frac{i}{2}(x-x')(t)N(t,t')(x-x')(t')\} \tag{2.3}$$

[2] We are labeling the degrees of freedom of the system and the environment by single letters x and q with the understanding that they can represent multiple or even infinite degrees of freedom, e.g. corresponding to a field.

where D and N stand for the real dissipation and noise kernels respectively. Note that in this quadratic order approximation, the influence action $S_{IF}(x, x')$ is related to the closed-time-path(CTP) or in-in effective action (for details on the CTP effective action see [31]) $\Gamma_{CTP}[x, x']$ through [50]

$$\Gamma_{CTP}[x, x'] = S[x] - S[x'] + S_{IF}[x, x']. \tag{2.4}$$

The equation of motion obtained from the CTP effective action for the expectation values are clearly seen to be real and causal [31]. They read

$$\frac{\delta}{\delta x(t)} \Gamma_{CTP}[x, x'] \bigg|_{x'=x=\bar{x}} = 0 \tag{2.5}$$

From the influence functional a Langevin equation for the system dynamics may be derived by a formal procedure, first introduced by Feynman and Vernon [33], which consists of introducing a Gaussian stochastic source $\xi(t)$ with $< \xi(t) >_\xi = 0$ and $< \xi(t)\xi(t') >= N(t, t')$ and defining an improved or stochastic effective action as

$$S_{eff}[x, x'; \xi] = S_s[x] - S_s[x'] + \mathcal{R}S_{IF}[x, x'] + \xi.(x - x'). \tag{2.6}$$

such that $\left\langle e^{iS_{eff}[x,x';\xi]} \right\rangle_\xi = e^{i\Gamma_{CTP}[x,x']}$. This leads to equations of motion with a stochastic force:

$$\frac{\delta S_{eff}[x, x'; \xi]}{\delta x} \bigg|_{x=x'} = 0. \tag{2.7}$$

or, equivalently,

$$\frac{\delta \Gamma_{CTP}[x, x'; \xi]}{\delta x} \bigg|_{x=x'} = 0. \tag{2.8}$$

The equation of motion obtained from (2.7) using (2.6) is

$$\frac{\partial S_s}{\partial x(t)} + \int dt' \; \gamma(t, t') \frac{d\, x(t')}{dt'} = \xi \tag{2.9}$$

where $D(t, t') = -\partial_{t'} \gamma(t, t')$. Being now in the form of a Langevin equation , the physical meaning of the γ and N kernels in Eq. (2.9) become clearer. Both the terms involving γ and ξ represent the backreaction of the environment on the system. However, γ (or more properly the odd part of γ) is associated with dissipation and ξ is a stochastic noise term associated with random fluctuations of the system exactly as the terms are interpreted in the context of Brownian motion. Averaging (2.9) over the noise using the appropriate probability distribution will give the semiclassical equation of motion for the mean value of x. The noise and dissipation originating from a closed system (as is done here, as opposed to being put in by hand) are in general related by a set of generalized fluctuation-dissipation relations, (FDR) which can be represented by a linear, non-local relation of the form,

$$N(t - t') = \int d(s - s')K(t - t', s - s')\gamma(s - s') \tag{2.10}$$

We will discuss various aspects of the FDR in greater detail in later sections. To keep the discussion simple, we have written the noise and dissipation kernels in terms of single scalar functions. However, the method is general enough to encompass multiple noise and dissipation kernels and cases where the kernels are tensorial, as in our later examples.

2.2 Stochastic Semiclassical Gravity

The framework described in the previous section is general enough to encompass any situation where we want to describe the semiclassical dynamics of the "system" degrees of freedom with the backreaction of the "environment" degrees of freedom incorporated self-consistently. One has to make sure there is a discrepancy parameter in the problem to define our interested subsystem and justify separating it from its environment. Coarse-graining the environment and incorporating its effect on the subsystem turns it into an open system. In the context of semiclassical gravity our "system" of interest is a spacetime metric coupled to the "environment" of a quantum scalar field (the discrepancy parameter being the Planck mass). The dynamic classical spacetime metric creates particles of the quantum field and these in turn provide a backreaction on the space time metric to alter its dynamics in response. This is captured in the so called semiclassical Einstein equations (SEE) which take the form

$$G_{\mu\nu}(g_{\alpha\beta}) = \kappa\langle T_{\mu\nu}\rangle_q \tag{2.11}$$

where $T_{\mu\nu}$ is the energy momentum tensor of, say, a free scalar field Φ, $G_{\mu\nu}$ is the Einstein tensor, $\kappa = 8\pi G_N$, G_N being the Newton's constant. Here $\langle\,\rangle_q$ denotes expectation value taken with respect to some quantum state with symmetry commensurate with that of the background spacetime. Studies of semiclassical Einstein equation have been carried out in the last two decades by many authors for cosmological [4] and black hole spacetimes [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In the analogy with the open system dynamics described in Section 2.1: Eq.(2.11) is equivalent to Eq.(2.5) where the degrees of freedom x of the system are identified with the metric $g_{\alpha\beta}$ and those of the environment q are identified with the scalar field $\Phi(x)$. However, from the discussion in the last section it is also clear that Eq. (2.5) , and hence also the semiclassical Einstein Eq. (2.11) results on averaging the full Langevin-type Eq. (2.7) over noise. Thus the semiclassical Einstein equation incorporates the dissipation but misses out the fluctuation aspect of the backreaction. The recognition of this crucial point [51] ushered in a new theory known as stochastic semiclassical gravity (SSG) , (or stochastic gravity in short as there is no confusion in this context as to where the stochasticity originates). Aided by the concept of open systems and the techniques of influence functional and the CTP effective action, stochastic gravity is the new framework where one should consider backreaction problem. We shall do this for the black hole problem here. Stochastic gravity can treat noise and fluctuations from particle creation on the same footing as dissipation. Spacetime dynamics is now governed by a stochastic generalization of the semiclassical Einstein equation known as the Einstein-Langevin equation, the analog of Eq. (2.9) in the context of semiclassical gravity. The conventional theory of semiclassical gravity (SCG) with sources given by the vacuum expectation value of the energy momentum tensor is viewed as a mean field approximation of this new theory. Schematically the Einstein-Langevin equation takes on the form

$$\begin{aligned} \tilde{G}_{\mu\nu}(x) &= \kappa\left(T^{\rm c}_{\mu\nu} + T^{\rm qs}_{\mu\nu}\right), \\ T^{\rm qs}_{\mu\nu} &\equiv \langle T_{\mu\nu}\rangle_{\rm q} + T^{\rm s}_{\mu\nu} \end{aligned} \tag{2.12}$$

Here, $T^{c}_{\mu\nu}$ is due to classical matter or fields, $\langle T_{\mu\nu}\rangle_q$ is the vacuum expectation value of the stress tensor of the quantum field, and $T^{\rm qs}_{\mu\nu}$ is a new stochastic term which is related to the fluctuations of $T_{\mu\nu}$ in the vacuum state. Taking the average of (2.12) with respect to the noise distribution will lead to the conventional semiclassical Einstein equation. It is clear in this context why SCG is regarded as a mean field theory. For a detailed derivation and solution of the Einstein-Langevin equation in the context of semiclassical cosmology the reader is referred to [56].

3 Black Holes: Backreaction and Fluctuations

Let us now come to the focus of this article, namely the issue of how to incorporate the new stochastic features of semiclassical gravity in the context of black holes. Our interest is to employ the full force of the formalism developed in section 2.1 to this problem and see the effect of the stochastic noise term on black hole backreaction. In this article we will sketch the strategy of this program rather than describe detailed calculations which are still in progress. We focus on the simpler class of problems of a quasi-static black hole in quasi-equilibrium (a box is required) with its Hawking radiation described by a scalar field. The goal is to obtain an influence action analogous to (2.6) for this model of a black hole coupled to a scalar field and to be able to derive an Einstein-Langevin equation analogous to (2.9) from it. To this end we will first proceed to derive the CTP effective action Γ_{CTP} for this case and then use it via (2.4) to derive the stochastic influence action (2.6).

Let us make a modest beginning by considering the simplest model of this class described by a perturbed Schwarzschild metric. This has been previously used by York [7] to analyze black hole backreaction. We do not intend to offer a *better* solution than what was done before, but to bring out the *new* physics unbeknown to researchers of backreaction problems in SCG, in particular, noise and fluctuations and their consequences.

3.1 The Model

In this model the black hole spacetime is described by a spherically symmetric static metric with line element of the following general form written in advanced time Eddington-Finkelstein coordinates

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\psi}\left(1 - \frac{2m}{r}\right)dv^2 + 2e^{2\psi}dvdr + r^2\ d\Omega^2 \tag{3.1}$$

where $\psi = \psi(r)$ and $m = m(r)$, $v = t + r + 2Mln\left(\frac{r}{2M} - 1\right)$ and $d\Omega^2$ is the line element on the two sphere. Hawking radiation is described by a massless, conformally coupled quantum scalar field ϕ with the classical action

$$S_m[\phi, g_{\mu\nu}] = -\frac{1}{2}\int d^n x\sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi(n)R\phi^2] \tag{3.2}$$

where $\xi(n) = \frac{(n-2)}{4(n-1)}$ (n is the dimension of spacetime) and R is the curvature scalar of the spacetime it lives in.

Let us consider linear perturbations $h_{\mu\nu}$ off a background Schwarzschild metric metric $g^{(0)}_{\mu\nu}$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \tag{3.3}$$

with standard line element

$$(ds^2)^0 = \left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{3.4}$$

We look for this class of perturbed metrics in the form given by (3.1), (thus restricting our consideration only to spherically symmetric perturbations):

$$e^{\psi} \simeq 1 + \epsilon\rho(r) \tag{3.5}$$

and

$$m \simeq M[1 + \epsilon\mu(r)] \tag{3.6}$$

where $\frac{\epsilon}{\lambda M^2} = \frac{1}{3}aT_H^4$; $a = \frac{\pi^2}{30}$; $\lambda = 90(8^4)\pi^2$. T_H is the Hawking temperature. This particular parametrization of the perturbation is chosen following York's [7] notation. Thus the only non-zero components of $h_{\mu\nu}$ are

$$h_{vv} = -\left((1 - \frac{2M}{r})2\epsilon\rho(r) + \frac{2M\epsilon\mu(r)}{r}\right) \tag{3.7}$$

and

$$h_{vr} = \epsilon\rho(r) \tag{3.8}$$

So this represents a metric with small static and radial perturbations about a Schwarzschild black hole. The initial quantum state of the scalar field is taken to be the Hartle Hawking vacuum, which is essentially a thermal state at the Hawking temperature and it represents a black hole in (unstable) thermal equilibrium with its own Hawking radiation.

Before we go on to describe our strategy for treating backreaction in this model, let us briefly recall York's analysis so that it will facilitate later comparisons with our program. York [7] [3] essentially considers the semiclassical Einstein equation

$$G_{\mu\nu}(g_{\alpha\beta}) = \kappa\langle T_{\mu\nu}\rangle \tag{3.9}$$

with $G_{\mu\nu} \simeq G^{(0)}_{\mu\nu} + \delta G_{\mu\nu}$ where $G^{(0)}_{\mu\nu}$ is the Einstein tensor for the background spacetime. The zeroth order solution gives a background metric in empty space, i.e, the Schwarzschild metric. $\delta G_{\mu\nu}$ is the linear correction to the Einstein tensor in the perturbed metric. The semiclassical Einstein equation in this approximation therefore reduces to

$$\delta G_{\mu\nu}(g^{(0)}, h) = \kappa < T_{\mu\nu} > \tag{3.10}$$

He proceeds to solve this equation to first order by using the expectation value of the energy momentum tensor for a conformally coupled scalar field in the Hartle-Hawking vacuum in the unperturbed (Schwarzschild) spacetime on the right hand side and $\delta G_{\mu\nu}$ on the left hand side is calculated using (3.7) and (3.8). Unfortunately, no exact analytical expression is available for the $< T_{\mu\nu} >$ in a Schwarzschild metric with the quantum field in the Hartle-Hawking vacuum that goes on the right hand side. York therefore uses the approximate expression given by Page [19] which is known to give excellent agreement with numerical results. Page's approximate expression for $< T_{\mu\nu} >$ was constructed using a thermal Feynman Green's function obtained by a conformal transformation of a WKB approximated Green's function for an optical Schwarzschild metric. York then solves the semiclassical Einstein equation (3.10) self consistently to obtain the corrections to the background metric induced by the backreaction encoded in the functions $\mu(r)$ and $\rho(r)$. There was no mention of fluctuations or its effects. As we shall see, in the language of the previous section, the semiclassical gravity procedure which York followed working at the equation of motion level, is equivalent to looking at the noise-averaged backreaction effects.

[3] See also work by Hochberg and Kephart [8] for a massless vector field, Hochberg, Kephart and York [9] for a massless spinor field, and Anderson, Hiscock, Whitesell, and York [10] for a quantized massless scalar field with arbitrary coupling to spacetime curvature

3.2 CTP Effective Action for the Black Hole

In this section, we set up the CTP effective action for the model described in the previous section. The treatment given in this section closely follows reference [45]. Using the metric (3.4) (and neglecting the surface terms that appear in an integration by parts) we have the action for the scalar field written perturbatively as

$$S_m[\phi, h_{\mu\nu}] \;=\; \frac{1}{2}\int d^n x\sqrt{-g^{(0)}}\;\phi\left[\Box + V^{(1)} + V^{(2)} + \cdots\right]\phi, \tag{3.11}$$

where the first and second order perturbative operators $V^{(1)}$ and $V^{(2)}$ are given by

$$\begin{aligned} V^{(1)} &\equiv -\frac{1}{\sqrt{-g^{(0)}}}\left\{[\partial_\mu\left(\sqrt{-g^{(0)}}\bar{h}^{\mu\nu}(x)\right)]\partial_\nu + \bar{h}^{\mu\nu}(x)\partial_\mu\partial_\nu + \xi(n)R^{(1)}(x)\right\}, \\ V^{(2)} &\equiv -\frac{1}{\sqrt{-g^{(0)}}}\left\{[\partial_\mu\left(\sqrt{-g^{(0)}}\hat{h}^{\mu\nu}(x)\right)]\partial_\nu + \hat{h}^{\mu\nu}(x)\partial_\mu\partial_\nu - \xi(n)(R^{(2)}(x) + \frac{1}{2}h(x)R^{(1)}(x))\right\} \end{aligned} \tag{3.12}$$

In the above expressions, $R^{(k)}$ is the k-order term in the pertubation $h_{\mu\nu}(x)$ of the scalar curvature R and $\bar{h}_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ denote a linear and a quadratic combination of the perturbation, respectively,

$$\begin{aligned} \bar{h}_{\mu\nu} &\equiv h_{\mu\nu} - \frac{1}{2}hg^{(0)}_{\mu\nu}, \\ \hat{h}_{\mu\nu} &\equiv h_\mu^{\;\alpha}h_{\alpha\nu} - \frac{1}{2}hh_{\mu\nu} + \frac{1}{8}h^2 g^{(0)}_{\mu\nu} - \frac{1}{4}h_{\alpha\beta}h^{\alpha\beta}g^{(0)}_{\mu\nu}. \end{aligned} \tag{3.13}$$

From quantum field theory in curved spacetime considerations we take the following action for the gravitational field (see [45, 43] for more details)

$$\begin{aligned} S_g^{(div)}[g_{\mu\nu}] &= \frac{1}{\ell_P^{n-2}}\int d^n x\,\sqrt{-g}R(x) \\ &\quad + \frac{\alpha\bar{\mu}^{n-4}}{4(n-4)}\int d^n x\,\sqrt{-g}\left[3R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(x) - \left(1 - 360(\xi(n) - \frac{1}{6})^2\right)R(x)R(x)\right] \end{aligned} \tag{3.14}$$

The first term is the classical Einstein-Hilbert action and the second term is the counterterm in four dimensions used to renormalize the divergent effective action. In this action $\ell_P^2 = 16\pi G$, $\alpha = (2880\pi^2)^{-1}$ and $\bar{\mu}$ is an arbitrary mass scale.

We are interested in computing the CTP effective action for the model given by the form (3.11) for the matter action and when the field Φ is initially in the Hartle- Hawking vacuum,. This is equivalent to saying that the initial state of the field is described by a thermal density matrix at a finite temperature $T = T_H$. The CTP effective action at finite temperature $T \equiv 1/\beta$ for this model is given by (for details see [45])

$$\Gamma^\beta_{CTP}[h^\pm_{\mu\nu}] \;=\; S_g^{(div)}[h^+_{\mu\nu}] - S_g^{(div)}[h^-_{\mu\nu}] - \frac{i}{2}Tr\{\ln \bar{G}^\beta_{ab}[h^\pm_{\mu\nu}]\}, \tag{3.15}$$

where $\pm$ denote the forward and backward time path of the CTP formalism and $\bar{G}^\beta_{ab}[h^\pm_{\mu\nu}]$ is the complete 2×2 matrix propagator (a and b take $\pm$ values: G_{++}, G_{+-} and G_{--} correspond to the Feynman, Wightman and Schwinger Greens functions respectively) with thermal boundary conditions for the differential operator $\sqrt{-g^{(0)}}(\Box + V^{(1)} + V^{(2)} + \cdots)$. The actual form of $\bar{G}^{\beta}_{ab}$ cannot be explicitly given. However, it is easy to obtain a perturbative expansion in terms of $V^{(k)}_{ab}$, the k-order matrix version of the complete differential operator defined by $V^{(k)}_{\pm\pm} \equiv \pm V^{(k)}_{\pm}$ and $V^{(k)}_{\pm\mp} \equiv 0$, and G^{β}_{ab}, the thermal matrix propagator for a massless scalar field in Schwarzschild spacetime . To second order $\bar{G}^{\beta}_{ab}$ reads,

$$\bar{G}^{\beta}_{ab} = G^{\beta}_{ab} - G^{\beta}_{ac}V^{(1)}_{cd}G^{\beta}_{db} - G^{\beta}_{ac}V^{(2)}_{cd}G^{\beta}_{db} + G^{\beta}_{ac}V^{(1)}_{cd}G^{\beta}_{de}V^{(1)}_{ef}G^{\beta}_{fb} + \cdots \tag{3.16}$$

Expanding the logarithm and dropping one term independent of the perturbation $h^{\pm}_{\mu\nu}(x)$, the CTP effective action may be perturbatively written as,

$$\begin{aligned}\Gamma^{\beta}_{CTP}[h^{\pm}_{\mu\nu}] \quad = \quad & S^{div}_{g}[h^{+}_{\mu\nu}] - S^{div}_{g}[h^{-}_{\mu\nu}] \\ & + \frac{i}{2}Tr[V^{(1)}_{+}G^{\beta}_{++} - V^{(1)}_{-}G^{\beta}_{--} + V^{(2)}_{+}G^{\beta}_{++} - V^{(2)}_{-}G^{\beta}_{--}] \\ & - \frac{i}{4}Tr[V^{(1)}_{+}G^{\beta}_{++}V^{(1)}_{+}G^{\beta}_{++} + V^{(1)}_{-}G^{\beta}_{--}V^{(1)}_{-}G^{\beta}_{--} - 2V^{(1)}_{+}G^{\beta}_{+-}V^{(1)}_{-}G^{\beta}_{-+}].\end{aligned} \tag{3.17}$$

However, unlike the case of [45] where $h_{\mu\nu}$ represented a perturbation about flat space and hence one had knowledge of exact "unperturbed" thermal propagators, in this case, since the perturbation is about Schwarzschild spacetime, exact expressions for the corresponding unperturbed propagators $G^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$ are not known. Therefore apart from the approximation of computing the CTP effective action to certain order in perturbation theory, an appropriate approximation scheme for the unperturbed Green's functions is also required. This feature manifested itself in York's calculation of backreaction as well, where in writing the $< T_{\mu\nu} >$ on the right hand side of the semiclassical Einstein equation in the unperturbed Schwarzschild metric, he had to use an approximate expression for $< T_{\mu\nu} >$ in the Schwarzschild metric given by Page [19]. The additional complication here is that while to obtain $< T_{\mu\nu} >$ as in York's calculation, the knowledge of only the thermal Feynman Green's function is required, to calculate the CTP effective action one needs the knowledge of the full matrix propagator, which involves the Feynman, Schwinger and Wightman functions.

It is indeed possible to construct the full thermal matrix propagator $G^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$ based on Page's approximate Feynman Green's function by using identities relating the Feynman Green's function with the other Green's functions with different boundary conditions. One can then proceed to explicitly compute a CTP effective action and the influence functional based on this approximation. However, we desist from delving into such a calculation for the following reason. Our main interest in performing such a calculation is to identify and analyze the noise term which is the new ingredient in the backreaction. We have mentioned that the noise term gives a stochastic contribution $T^{\rm s}_{\mu\nu}$ to the Einstein Langevin equation (2.12). We had also stated that this term is related to the variance of fluctuations in $T_{\mu\nu}$, i.e, schematically, to $< T^2_{\mu\nu} >$. However, a calculation of $< T^2_{\mu\nu} >$ in the Hartle-Hawking state in a Schwarzschild background using the Page approximation was performed by Phillips and Hu [46, 47, 48] and it was shown that though the approximation is excellent as far as $< T_{\mu\nu} >$ is concerned, it gives unacceptably large errors for $< T^2_{\mu\nu} >$ at the horizon. In fact, similar errors will be propagated in the non-local dissipation term as well, because both terms originate from the same source, that is, they come from the last trace term in (3.17) which contains terms quadratic in the Green's function. However, the Influence Functional or CTP formalism itself does not depend on the nature of the approximation, so we will attempt to exhibit the general structure of the calculation without resorting to a specific form for the Greens function and conjecture on what is to be expected. A more accurate computation can be performed using this formal structure once a better approximation becomes available.

If we denote the difference and the sum of the perturbations $h^{\pm}_{\mu\nu}$ by $[h_{\mu\nu}] \equiv h^{+}_{\mu\nu} - h^{-}_{\mu\nu}$ and $\{h_{\mu\nu}\} \equiv h^{+}_{\mu\nu} + h^{-}_{\mu\nu}$, respectively, the influence functional form of the thermal CTP effective action may be written to second order in $h_{\mu\nu}$ as [45],

$$
\begin{aligned}
\Gamma^{\beta}_{CTP}[h^{\pm}_{\mu\nu}] \quad \simeq \quad & \frac{1}{2\ell_P^2}\int d^4x\ d^4x'\ [h_{\mu\nu}](x)L^{\mu\nu,\alpha\beta}_{(o)}(x,x')\{h_{\alpha\beta}\}(x') \\
& +\frac{1}{2}\int d^4x\ [h_{\mu\nu}](x)T^{\mu\nu}_{(\beta)} \\
& +\frac{1}{2}\int d^4x\ d^4x'\ [h_{\mu\nu}](x)H^{\mu\nu,\alpha\beta}(x,x')\{h_{\alpha\beta}\}(x') \\
& -\frac{1}{2}\int d^4x\ d^4x'\ [h_{\mu\nu}](x)D^{\mu\nu,\alpha\beta}(x,x')\{h_{\alpha\beta}\}(x') \\
& +\frac{i}{2}\int d^4x\ d^4x'\ [h_{\mu\nu}](x)N^{\mu\nu,\alpha\beta}(x,x')[h_{\alpha\beta}](x').
\end{aligned}
\tag{3.18}
$$

The first line is the Einstein-Hilbert action to second order in the perturbation $h^{\pm}_{\mu\nu}(x)$ and $\mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x)$ is a symmetric kernel, *i.e.* $\mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x,x') = \mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x',x)$. The second is a local term linear in $h^{\pm}_{\mu\nu}(x)$. $T^{\mu\nu}_{(\beta)}$ represents the zeroth order contribution to $< T_{\mu\nu} >$ and far away from the hole it takes the form of the stress tensor of massless scalar particles at temperature β^{-1}. The third and fourth terms constitute the remaining quadratic component of the real part of the effective action. The kernels $H^{\mu\nu,\alpha\beta}(x,x')$ and $D^{\mu\nu,\alpha\beta}(x,x')$ are respectively even and odd in x,x'. The last term gives the imaginary part of the effective action and the kernel $N(x,x')$ is symmetric. This is the general structure of the CTP effective action arising from the calculation of the traces in equation (3.17). Of course, to write down explicit expressions for the non-local kernels one requires the input of the explicit form of $G^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$, which we have not used. In spite of this limitation we can make some interesting observations from this effective action. Connecting this thermal CTP effective action to the influence functional via equation (2.4) we see that the nonlocal imaginary term containing the kernel $N^{\mu\nu,\alpha\beta}(x,x')$ is responsible for the generation of the stochastic noise term in the Einstein-Langevin equation and the real non-local term containing kernel $D^{\mu\nu,\alpha\beta}(x,x')$ is responsible for the non-local dissipation term. The Einstein-Langevin equation can be generated from equation (2.8) by first constructing the improved semiclassical effective action in accordance with (2.6) and deriving the equation of motion (2.7) by taking a functional derivative of the above effective action with respect to $[h_{\mu\nu}]$ and equating it to zero. With the identification of noise and dissipation kernels, one can use them to write down a Fluctuation-Dissipation relation (FDR) analogous to (2.10) in the context of black holes.

3.3 Einstein-Langevin equation

In this section we show how a semiclassical Einstein-Langevin equation can be derived from the previous thermal CTP effective action. This equation depicts the stochastic evolution of the perturbations of the black hole under the influence of the fluctuations of the thermal scalar field.

The influence functional $\mathcal{F} \equiv \exp(iS_{IF})$ previously introduced in section 2.1 can be written in terms of the the CTP effective action $\Gamma^{\beta}_{CTP}[h^{\pm}_{\mu\nu}]$ derived in equation (3.18) as follows using the connection given by equation (2.4)

$$\mathcal{F} \;=\; \exp i\left(Re\{\Gamma^{\beta}_{CTP}[h^{\pm}_{\mu\nu}]\} + \frac{i}{2}\int d^4x\; d^4x'\;[h_{\mu\nu}](x)\mathrm{N}^{\mu\nu,\alpha\beta}(x-x')[h_{\alpha\beta}](x')\right), \tag{3.19}$$

where $Re\{\ \}$ denotes taking the real part. Following [33, 52], we can interpret the real part of the influence functional as the characteristic functional of a non-dynamical stochastic variable $j^{\mu\nu}(x)$,

$$\Phi([h_{\mu\nu}]) \;=\; \exp\left(-\frac{1}{2}\int d^4x\; d^4x'\;[h_{\mu\nu}](x)\mathrm{N}^{\mu\nu,\alpha\beta}(x-x')[h_{\alpha\beta}](x')\right). \tag{3.20}$$

This classical stochastic field represents probabilistically the quantum fluctuations of the matter field and is responsible for the dissipation of the gravitational field. By definition, the above characteristic functional is the functional Fourier transform of the probability distribution functional $\mathcal{P}[j^{\mu\nu}]$ with respect to $j^{\mu\nu}$,

$$\Phi([h_{\mu\nu}]) \;=\; \int \mathcal{D}j^{\mu\nu}\;\mathcal{P}[j^{\mu\nu}]\;e^{i\int d^4x\;[h_{\mu\nu}](x)j^{\mu\nu}(x)}. \tag{3.21}$$

Using (3.20) one can easily see that the probability distribution functional is related with the noise kernel by the formal expression,

$$\mathcal{P}[j^{\mu\nu}] \;=\; \frac{\exp\left(-\frac{1}{2}\int d^4x\; d^4x'\; j_{\mu\nu}(x)[\mathrm{N}^{\mu\nu,\alpha\beta}(x-x')]^{-1}j_{\alpha\beta}(x')\right)}{\int \mathcal{D}j^{\mu\nu}\;\exp\left(-\frac{1}{2}\int d^4x\; d^4x'\; j_{\mu\nu}(x)[\mathrm{N}^{\mu\nu,\alpha\beta}(x-x')]^{-1}j_{\alpha\beta}(x')\right)}. \tag{3.22}$$

For an arbitrary functional of the stochastic field $\mathcal{A}[j^{\mu\nu}]$, the average value with respect to the previous probability distribution functional is defined as the functional integral

$$\langle\mathcal{A}[j^{\mu\nu}]\rangle_j \equiv \int \mathcal{D}[j^{\mu\nu}]\;\mathcal{P}[j^{\mu\nu}]\mathcal{A}[j^{\mu\nu}]. \tag{3.23}$$

In terms of this stochastic average the influence functional can be written as $\mathcal{F} = \langle\exp\left(i\Gamma^{st}_{CTP}[h^{\pm}_{\mu\nu}]\right)\rangle_j$, where $\Gamma^{st}_{CTP}[h^{\pm}_{\mu\nu}]$ is the stochastic effective action

$$\Gamma^{st}_{CTP}[h^{\pm}_{\mu\nu}] \;\equiv\; Re\{\Gamma^{\beta}_{CTP}[h^{\pm}_{\mu\nu}]\} + \int d^4x\;[h_{\mu\nu}](x)j^{\mu\nu}(x). \tag{3.24}$$

Clearly, because of the quadratic definition of the characteristic functional (3.20) and its relation with the probability distribution functional (3.21), the field $j^{\mu\nu}(x)$ is a zero mean Gaussian stochastic variable. This means that its two-point correlation function, which is given in terms of the noise kernel by

$$\langle j^{\mu\nu}(x)j^{\alpha\beta}(x')\rangle_j \;=\; \mathrm{N}^{\mu\nu,\alpha\beta}(x-x'), \tag{3.25}$$

completely characterizes the stochastic process. The Einstein-Langevin equation follows from taking the functional derivative of the stochastic effective action (3.24) with respect to $[h_{\mu\nu}](x)$ and imposing $[h_{\mu\nu}](x)=0$ [52] as an analog of (2.7). In our case, this leads to

$$\frac{1}{\ell_P^2}\int d^4x'\;\mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x-x')h_{\alpha\beta}(x')+\frac{1}{2}\,T^{\mu\nu}_{(\beta)}+\int d^4x'\;\left(\mathrm{H}^{\mu\nu,\alpha\beta}(x-x')-\mathrm{D}^{\mu\nu,\alpha\beta}(x-x')\right)h_{\alpha\beta}(x')+j^{\mu\nu}(x) \;=\; 0. \tag{3.26}$$

In the far field limit this equation should reduce to that obtained by Campos and Hu [45]: For gravitational perturbations $h^{\mu\nu}$ defined in (3.13) under the harmonic gauge $\bar{h}^{\mu\nu}{}_{,\nu} = 0$, their Einstein-Langevin equation is given by

$$\Box \bar{h}^{\mu\nu}(x) + \ell_P^2 \left\{ T^{\mu\nu}_{(\beta)} + 2P_{\rho\sigma,\alpha\beta} \int d^4x' \, \left(\mathrm{H}^{\mu\nu,\alpha\beta}(x-x') - \mathrm{D}^{\mu\nu,\alpha\beta}(x-x') \right) \bar{h}^{\rho\sigma}(x') + 2j^{\mu\nu}(x) \right\} = 0, \tag{3.27}$$

where the tensor $P_{\rho\sigma,\alpha\beta}$ is given by

$$P_{\rho\sigma,\alpha\beta} \;=\; \frac{1}{2} \left(\eta_{\rho\alpha}\eta_{\sigma\beta} + \eta_{\rho\beta}\eta_{\sigma\alpha} - \eta_{\rho\sigma}\eta_{\alpha\beta} \right). \tag{3.28}$$

The expression for $P_{\rho\sigma,\alpha\beta}$ in the near horizon limit of course cannot be expressed in such a simple form. Note that this differential stochastic equation includes a non-local term responsible for the dissipation of the gravitational field and a noise source term which accounts for the fluctuations of the quantum field . Note also that this equation in combination with the correlation for the stochastic variable (3.25) determine the two-point correlation for the stochastic metric fluctuations $\langle \bar{h}_{\mu\nu}(x)\bar{h}_{\alpha\beta}(x') \rangle_j$ self-consistently.

4 CTP Effective Action in terms of Bogolyubov Coefficients

Before we draw further implications from this structural form of the CTP effective action and the stochastic effective action on the effect of fluctuations and dissipation, we would like to add another perspective to this problem by considering a different aspect of the interpretation of the stochastic fluctuation term, namely its connection with the fluctuation in the number of particles produced. Studies of the Influence Functional in the context of semiclassical cosmology [60, 61, 59, 49] show that the Influence Functional can be written in terms of the Bogolyubov coefficients that represent particle production in the specific cosmological background. The noise kernel that arises in this Influence functional can be related to the difference between the number of particles produced along alternative histories. The noise term in the Einstein-Langevin equation can in turn be related to the fluctuations in the number of particles produced by the cosmological background. The relationship between noise and particle number fluctuations has also been emphasized in [62]. Gaining insight from this approach, we have attempted to cast the Influence Functional or equivalently the CTP effective action for the black hole in a similar form in terms of Bogolyubov coeffecients. In this case these coefficients refer to the production of particles by the black hole background. In this section we will describe such a derivation.

The CTP effective action, which is a functional of two metric histories, $g^{(+)}_{\mu\nu}$ and $g^{(-)}_{\mu\nu}$, may be expressed in terms of the Bogolyubov coefficients that relate the "in" vacuum state to the "out" vacuum state in these two histories. Such a representation has been derived earlier [59, 63] for cosmological metrics. For these cases, the Bogolyubov coefficient matrices (denoted by α and β) are diagonal. In the general case (and specifically for black hole metrics), the Bogolyubov coefficient matrices are not diagonal, and the derivation of the CTP effective action is a bit more involved.

Here, we follow the vacuum-to-many-particle-amplitude approach of DeWitt [64], who used this method to derive an expression for the "in-out" effective action in terms of Bogolyubov coefficients.

The CTP generating functional is the product of the time-evolution amplitude from a vacuum state in the distant past ("in" vacuum) to an arbitrary state in the distant future in the metric

$g_{\mu\nu}^{(+)}$ with the time-reversed amplitude from the same arbitrary state in the distant future to the "in" vacuum state in the metric $g_{\mu\nu}^{(-)}$, summed over all arbitrary states in the distant future. We thus obtain

$$e^{i\Gamma_{\rm CTP}} = \sum_{n,i_1,...i_n} \frac{1}{n!} \, {}_{\rm in}\langle 0 \mid 1_{i_1}, 1_{i_2}, ...1_{i_n}\rangle^{(+)}_{\rm out} \, {}_{\rm out}\langle 1_{i_1}, 1_{i_2}, ..1_{i_n} \mid 0\rangle^{(-)}_{\rm in}, \tag{4.1}$$

where $\mid 1_{i_1}, 1_{i_2}, ...1_{i_n}\rangle_{\rm out}$ is the state containing one particle each in modes $i_1, i_2, \ldots, i_n$ in the "out" particle model, the $(+)$ and $(-)$ superscripts refer to the two metric histories, and the $n!$ in the denominator avoids overcounting of states.

DeWitt [64] writes down the amplitude to evolve from the "in" vacuum state to the state containing single particles in m modes in the distant future, in terms of the Bogolyubov coefficients, as follows:

$${}_{\rm out}\langle 1_{i_1}, 1_{i_2}, ..1_{i_m} \mid 0\rangle^{(-)}_{\rm in} = e^{i\Gamma^{(-)}_{\rm in-out}} \sum_{n=0}^{\infty} \frac{i^{n/2}}{n!} \sum_{j_1,j_2,...,j_n} V^{(-)}_{j_1,...,j_n} {}_{\rm out}\langle 1_{i_1}, 1_{i_2}, ..., 1_{i_m} \mid 1_{j_1}, 1_{j_2}, ..., 1_{j_n}\rangle_{\rm out}, \tag{4.2}$$

where

$$V_{i_1,...,i_n} = \sum_p V_{i_1 i_2} \cdots V_{i_{n-1} i_n}, \quad n \text{ even} \tag{4.3}$$

$$= 0, \quad n \text{ odd} \tag{4.4}$$

with

$$V_{ij} = i \sum_k \beta^*_{ki} \alpha^{-1}_{jk}, \tag{4.5}$$

and the sum in Eq. (4.3) is over all $n!/(2^{n/2}(n/2)!)$ distinct pairings of the labels $i_1, ..., i_n$.

Note that

$${}_{\rm out}\langle 1_{i_1}, 1_{i_2}, ..., 1_{i_m} \mid 1_{j_1}, 1_{j_2}, ..., 1_{j_n}\rangle_{\rm out} = \delta_{mn} \sum_\gamma \delta_{i_1 j_{\gamma(1)}} \ldots \delta_{i_n j_{\gamma(n)}} \tag{4.6}$$

where the sum is now over all possible permutations γ of the indices $1, \ldots, n$. This allows us to simplify Eq. (4.2) to yield

$$\begin{aligned} {}_{\rm out}\langle 1_{i_1}, 1_{i_2} \ldots, 1_{i_m} \mid 0\rangle^{(-)}_{\rm in} &= e^{i\Gamma^{(-)}_{\rm in-out}} \frac{i^{m/2}}{m!} \sum_\gamma V_{i_{\gamma(1)} \ldots i_{\gamma(m)}} \\ &= e^{i\Gamma^{(-)}_{\rm in-out}} i^{m/2} V_{i_1 \ldots i_m}, \end{aligned} \tag{4.7}$$

where the second equality follows from the first one because each distinct pairing of indices has been repeated $m!$ times in the sum over all permutations.

Equation (4.7) and its complex conjugate in the $(+)$ history may be substituted in Eq. (4.1) to yield

$$\begin{aligned} e^{i\Gamma_{\rm CTP}} &= e^{i\Gamma^{(-)}_{\rm in-out}} e^{-i\Gamma^{(+)\,*}_{\rm in-out}} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1,...,i_n} V^{(-)}_{i_1,...,i_n} V^{(+)\,*}_{i_1,...,i_n} \\ &= (\det \alpha^{(-)})^{-1/2} (\det \alpha^{(+)*})^{-1/2} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1,...,i_n} V^{(-)}_{i_1,...,i_n} V^{(+)\,*}_{i_1,...,i_n}, \end{aligned} \tag{4.8}$$

where the representation of the in-out effective action in terms of the α Bogolyubov coefficient matrix has been derived in [64]. In order to evaluate the infinite sum in the above equations, we may expand the $V^{(-)}_{i_1,\ldots,i_n} V^{(+)*}_{i_1,\ldots,i_n}$ term using Eq. (4.3) and collect like powers of $V^{(-)}V^{(+)\dagger}$. We then observe that the general term in the infinite sum is of the form

$$\sum_{r=1}^{n} \frac{1}{2^r r!} \sum_{\{m_i \geq 1; \sum_{i=1}^{r} m_i = n\}} \frac{\mathrm{Tr}(V^{(-)}V^{(+)\dagger})^{m_1} \cdots \mathrm{Tr}(V^{(-)}V^{(+)\dagger})^{m_r}}{\prod_{i=1}^{r} m_i}, \tag{4.9}$$

where the second sum is over all possible values of r natural numbers $m_i, i = 1, \ldots, r$ whose sum is n.

One may readily verify that (4.9) is also the general term in the expansion of

$$\det(1 - V^{(-)}V^{(+)\dagger})^{-1/2} = \exp\left(-\frac{1}{2}\mathrm{Tr}\ln(1 - V^{(-)}V^{(+)\dagger})\right). \tag{4.10}$$

We thus obtain the CTP effective action in terms of Bogolyubov coefficients, as

$$\begin{aligned} e^{i\Gamma_{\mathrm{CTP}}} &= (\det\alpha^{(-)})^{-1/2}(\det\alpha^{(+)*})^{-1/2}\det(1 - V^{(-)}V^{(+)\dagger})^{-1/2} \\ &= \det\left(\alpha^{(+)*}\alpha^{(-)} - \beta^{(-)*}\beta^{(+)}\right)^{-1/2}, \end{aligned} \tag{4.11}$$

where the second equality follows from Eq. (4.5) and the use of $\det(AB) = \det(BA)$.

Note that the above effective action is the full unrenormalized CTP effective action and therefore the determinant is divergent. In practice, these divergences must be isolated and absorbed into the metric and the curvatures in the standard manner.

Our formula (4.11) reduces in a straightforward way to the formula for the cosmological case given in [59]. Note that in this section we have assumed the "in" vacuum state to be a pure state. Therefore, (4.11) does not directly apply to situations where the "in" state is a density matrix, as in, for example, the Hartle-Hawking vacuum state discussed in the previous section. Generalizations of (4.11) to arbitrary initial states (i.e., pure or mixed) follow a similar line of reasoning as discussed here and will be given in a future work. The restricted version of the CTP effective action given here can, however, be used to study evaporating black holes where the initial state is the Unruh vacuum, and to derive an Einstein-Langevin equation for the evaporating black hole.

The Bogolyubov coefficient representation may also serve as a seed for approximation schemes that are based on the Bogolyubov coefficients rather than on the Green's functions. Viewing the noise or fluctuation contribution in terms of fluctuations in particle number also offers the interesting possibility of interpretation in terms of the **isothermal compressibility of the vacuum** by exploiting the thermodynamic properties of black holes.

5 Discussions

At this point it is a good idea to take stock of what has been achieved by taking this strategy and also compare the results with earlier approaches. By setting up the CTP effective action/Influence Functional for this model system, we have a formal causal framework for dealing with the back-reaction of quantum fields on a black hole within the limitations of linearization and the special symmetries of the model. At present,though we know that with the Page approximation some backreaction terms in the Einstein-Langevin equation become unreliable, as and when better approximations to the Schwarzschild Green's functions become available this Influence Functional can be used to yield an Einstein-Langevin equation for black hole back reaction which encompasses the effects of dissipation as well as noise and fluctuations.

If we confine ourselves to Page's approximation and derive the equation of motion using (2.5) rather than (2.8), i.e, without the stochastic term, we expect to recover York's semiclassical Einstein's equation if one retains only the zeroth order contribution, i.e, the first two terms in the expression for the CTP effective action in equation (3.18) Thus, this offers a new route to arrive at York's semiclassical Einstein's equations. Not only is it a derivation of York's result from a different point of view, but it also shows how his result arises as an appropriate limit of a more complete framework, i.e, it arises when one averages over the noise. Another point worth noting is that our treatment will also yield a non-local dissipation term arising from the fourth term in equation (3.18) in the CTP effective action which is absent in York's treatment. This difference arises primarily due to the difference in the way backreaction is treated at the level of iterative approximations on the equation of motion versus the treatment in the effective action framework. In York's treatment, the Einstein tensor is computed to first order in perturbation theory , while $< T_{\mu\nu} >$ on the right hand side of the semiclassical Einstein equation is replaced by the zeroth order term. In the effective action treatment the full effective action is computed to second order in perturbation, and hence includes the higher order non-local terms.

The other important conceptual point that comes to light from this approach is that related to the Fluctuation-Dissipation Relation. Here it is clearly seen that the backreaction of quantum fields on black holes consists of two forms – dissipation and fluctuation or noise, which contribute respectively to the real and imaginary parts of the Influence functional as embodied in the dissipation and noise kernels. Again drawing the analogy with Brownian motion, these correspond to the dissipation of the energy of the Brownian particle as it approaches equilibrium, and the fluctuations at equilibrium. These are connected by the Fluctuation - Dissipation relation given by (2.10). Pursuing the analogy we conjecture that a FDR similar to (2.10) will also exist between the noise and dissipation kernels for the black hole case that we have described. This reveals an interesting connection between black holes interacting with quantum fields and non-equilibrium statistical mechanics. The existence of a FDR for the black hole case has been discussed by some authors previously [41, 42]. In [39] we have described in some detail how our approach differs from those of previous authors. We refer the reader to [39] for details.

There are, of course, limitations of our program. One obvious one is that we have to confine ourselves to small perturbations about the static Schwarzschild background. As a result we cannot hope to address questions about the fully dynamical collapse problem. However, it will allow us to study the stability of the black hole under the influence of stochastic fluctuations of the energy momentum tensor dictated by the noise terms.

Another limitation we have already discussed before is the problem of finding a reliable approximation to the Schwarzschild thermal Green's function to explicitly compute the noise and dissipation kernels. This limits our ability to present explicit analytical expressions for these kernels. One possibility is to try to work on improving on Page's approximation by retaining terms to higher order. A less ambitious first step could be to confine attention to the horizon and using approximations that are restricted to near the horizon and work out the Influence Functional in this regime. This is currently being pursued by us. The Influence Functional in the far field regime has already been worked out in [45].

The other shortcoming is the following. Though we have allowed for backreaction effects to modify the initial state in the sense that the temperature of the Hartle-Hawking state gets affected by the backreaction, we have essentially confined our analysis to a Hartle-Hawking thermal state of the field. This analysis does not directly extend to a more general class of states, for example to the case where the initial state of the field is in the Unruh vacuum. Thus we will not be able to comment on issues of the stability of an isolated radiating black hole under the influence of stochastic fluctuations.

It is also pertinent to mention the connection of our work with that of some other authors who have also pursued the viewpoint that the black hole metric is a stochastically fluctuating quantity and have studied its effect on Hawking radiation. For example, Casher et al [65] and Sorkin [66] have concentrated on the issue of fluctuations of the horizon induced by a fluctuating metric. Casher et al [65] considers the fluctuations of the horizon induced by the "atmosphere" of high angular momentum particles near the horizon, while Sorkin [66] calculates fluctuations of the shape of the horizon induced by the quantum field fluctuations under a Newtonian approximations. Both group of authors come to the conclusion that horizon fluctuations become large at scales much larger than the Planck scale (note Ford and Svaiter [69] later presented results contrary to this claim). However, though these works do deal with backreaction, the fluctuations considered do not arise as an explicit stochastic noise term as in our treatment. It may be worthwhile exploring the horizon fluctuations induced by the stochastic metric in our model and comparing the conclusions with the above authors. Barrabes et al [67] (also see [68] for work in similar vein) have considered the propagation of null rays and massless fields in a black hole fluctuating geometry and have shown that the stochastic nature of the metric leads to a modified dispersion relation and helps to confront the trans-Planckian frequency problem. However, in this case the stochastic noise is put in by hand and does not naturally arise from coarse graining as in the quantum open systems approach. It also does not take backreaction into account. It will be interesting to explore how a stochastic black hole metric, arising as a solution to the Einstein-Langevin equation, hence fully incorporating backreaction would affect the trans-Planckian problem.

Ford and his collaborators [69] have also explored the issue of metric fluctuations in detail and in particular have studied the fluctuations of the black hole horizon induced by metric fluctuations. However, the fluctuations they considered are in the context of a fixed background and do not relate to the backreaction.

Another work originating from the same vein of stochastic gravity but not complying with the backreaction spirit is that of Hu and Shiokawa [70], who study novel effects associated with electromagnetic wave propagation in a Robertson-Walker universe and the Schwarzschild spacetime with a small amount of given metric stochasticity. For the Schwarzschild metric, they find that time-independent randomness can decrease the total luminosity of Hawking radiation due to multiple scattering of waves outside the black hole and gives rise to event horizon fluctuations and fluctuations in the Hawking temperature. The stochasticity in a background metric in their work is assumed rather than derived (from quantum field fluctuations, as in this work) and so is not in the same spirit of backreaction. But it is interesting to compare their results with that of backreaction, so one can begin to get a sense of the different sources of stochasticity and their weights (see, e.g., [35] for a list of possible sources of stochasticity.)

In a subsequent paper Shiokawa [71] showed that the scalar and spinor waves in a stochastic spacetime behave similarly to the electrons in a disordered system. Viewing this as a quantum transport problem, he expressed the conductance and its fluctuations in terms of a nonlinear sigma model in the closed time path formalism and showed that the conductance fluctuations are universal, independent of the volume of the stochastic region and the amount of stochasticity. This result can have significant importance in characterizing the mesoscopic behavior of spacetimes resting between the semiclassical and the quantum regimes.

The stochastic approach to the study of black hole backreaction thus has a very rich structure and opens up many new avenues of inquiry. In particular it provides the proper platform and framework to launch a new program of research into the **nonequilibrium black hole thermodyamics**.

Acknowledgements BLH wishes to thank Toni Campos, Nicholas Phillips and K. Shiokawa for collaborations on topics related to black hole backreaction and fluctuations, and Drs. Paul Anderson, Larry Ford and Enric Verdaguer for general discussions on related problems. This work is supported in part by NSF grant PHY98-00967.

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